

THE DIFFUSION COEFFICIENT FOR PIECEWISE EXPANDING MAPS OF THE INTERVAL WITH METASTABLE STATES

Dmitry Dolgopyat* and Paul Wright†

November 25, 2010

Abstract

Consider a piecewise smooth expanding map of the interval possessing several invariant subintervals and the same number of ergodic absolutely continuous invariant probability measures (ACIMs). After this system is perturbed to make the subintervals lose their invariance in such a way that there is a unique ACIM, we show how to approximate the diffusion coefficient for an observable of bounded variation by the diffusion coefficient of a related continuous time Markov chain.

Key words: Expanding maps, absolutely continuous invariant measure, transfer operator, metastable states, slow dynamics.

AMS Subject Classification: 37D50, 60J28

*Dedicated to Manfred Denker on the occasion of his 60th birthday.*¹

1 Introduction

Metastable dynamics arise in a number of physical systems. In such systems, the phase space can be divided into a finite number of components, called metastable states, that are nearly invariant under the dynamics. A typical trajectory will remain in one metastable state for an extended period of time before escaping to another metastable state and repeating this behavior. Rigorous results about metastability in dynamical systems perturbed by noise can be found in [6] and [9].

*Department of Mathematics, University of Maryland, College Park, MD 20742. Email: dmitry@math.umd.edu. DD is partially supported by the NSF.

†Department of Mathematics, University of Maryland, College Park, MD 20742. Email: paulrite@math.umd.edu. PW is partially supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship.

¹We thank Carlangelo Liverani for useful comments on a preliminary version of this paper.

In this paper we are concerned with reducing the description of purely deterministic chaotic systems to coresponding finite state Markov chains. In particular, we continue the study of the dynamics of hyperbolic interval maps with metastable states initiated in [7]. These systems arise from perturbing an initial system T_0 with m disjoint invariant intervals I_1, I_2, \dots, I_m . The initial map has m mutually singular ergodic absolutely continuous invariant measures (ACIMs), $\mu_1, \mu_2, \dots, \mu_m$. T_0 is perturbed in such a way that the μ_j lose their invariance, and the perturbed map T_ε has only one ACIM, μ_ε . See Figure 1.

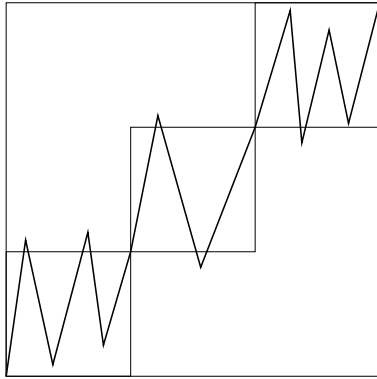


Figure 1: A map with three almost invariant intervals

Such metastable systems can be understood in the context of deterministic dynamical systems with holes (see [4]) as follows. As the invariance of initially invariant intervals is destroyed by the perturbation, we think of the small set of points $I_i \cap T_\varepsilon^{-1}I_j$ that switch from I_j to I_i after the application of T_ε , as being holes in the initially invariant sets. Therefore the techniques developed to study systems with holes are useful in our analysis.

As was shown in [7], we are able to approximate μ_ε , for small ε , by a convex combination $\sum_{j=1}^m p_j \mu_j$ of the initially invariant measures, where (p_1, p_2, \dots, p_m) is the invariant measure for the continuous time Markov chain on m states with transition rates proportional to the asymptotic sizes of the holes. Intuitively, since our map is chaotic on each interval, we expect the transition times between the I_j 's to be almost independent, which explains the appearance of the above mentioned Markov chain. In this paper, we show that the Markov approximation also extends to the diffusion matrix for smooth observables. We believe the methods developed in our paper can be used to describe the transport coefficients in other chaotic systems, for example, billiards with narrow tunnels [12]. However, in order to present the ideas of our proof in the simplest possible setting, we restrict our attention here to the setup of [7].

2 Statement of the main results

In this section, we define a family of dynamical systems with m nearly invariant (metastable) subsets. They are perturbations of a one-dimensional piecewise smooth expanding map with m invariant subintervals I_1, I_2, \dots, I_m of positive Lebesgue measure. On each of these intervals, the unperturbed system has a unique ACIM. The perturbations break this invariance in such a way that each perturbed system will have only one ACIM. Our main result is an asymptotic formula for the diffusion coefficient of a smooth observable as the size of the perturbation tends to zero. We also show that the sequence of jump times in between different intervals asymptotically approach exponential random variables.

Let $I = [0, 1]$. In this paper, a map $T : I \rightarrow I$ is called a piecewise C^2 map with $\mathcal{C} = \{0 = c_0 < c_1 < \dots < c_d = 1\}$ as a critical set if for each i , $T|_{(c_i, c_{i+1})}$ extends to a C^2 function on a neighborhood of $[c_i, c_{i+1}]$. We call T uniformly expanding if its minimum expansion, $\inf_{x \in I \setminus \mathcal{C}_0} |T'_0(x)|$, is greater than 1. As is customary for piecewise smooth maps, we consider T to be bi-valued at points $c_i \in \mathcal{C}$ where it is discontinuous. In such cases we let $T(c_i)$ be both values obtained as x approaches c_i from either side.

2.1 The initial system and its perturbations

The unperturbed system is a piecewise C^2 uniformly expanding map $T_0 : I \rightarrow I$ with $\mathcal{C}_0 = \{0 = c_{0,0} < c_{1,0} < \dots < c_{d,0} = 1\}$ as a critical set. There are boundary points $\mathcal{B} = \{b_j\} \subset (0, 1)$ such that $I_j = [b_{j-1}, b_j]$ ($b_0 = 0, b_m = 1$) i.e. $T_0(I_j) \subset I_j$. The existence of an ACIM of bounded variation for $T_0|_{I_j}$ is guaranteed by [10]. We assume in addition:

(I) *Unique mixing ACIMs on the initially invariant subsets:* $T_0|_{I_j}$, $j \in \{1 \dots m\}$, has only one ACIM μ_j , whose density is denoted by $\phi_j = d\mu_j/dx$. (T_0, μ_j) is mixing.

From (I), it follows that all ACIMs of T_0 are convex combinations of the ergodic ones, $\{\mu_j\}$.

We define the points in $H_0 = (T_0^{-1}\mathcal{B}) \setminus \mathcal{B}$ to be *infinitesimal holes*. (The exclusion of boundary points from the set of infinitesimal holes is not essential, although it does simplify our presentation. See assumption (V) and the discussion thereafter.)

(II) *No return of the critical set to the infinitesimal holes:* For every $k > 0$, $(T_0^k \mathcal{C}_0) \cap H_0 = \emptyset$.

Since ϕ_j are of bounded variation, they can be suitably defined so that they are continuous except on at most a countable set of points where it has jump discontinuities. Moreover (see [7, Section 4.2]), (II) implies that after ϕ_j have been so defined, they are continuous at each of the infinitesimal holes.

(III) *Positive ACIMs at infinitesimal holes:* ϕ_j is positive at each of the points in $H_0 \cap I_j$.

(II) and (III) are generic conditions, although (II) may be difficult to verify for specific examples. Similar assumptions are made elsewhere, including in [9, Section 3.2] and [7].

(IV) *Restriction on periodic critical points:* Either

(a) $\inf_{x \in I \setminus \mathcal{C}_0} |T'_0(x)| > 2$, or

(b) T_0 has no periodic critical points, except possibly that 0 or 1 may be fixed points.

Because T_0 may be bi-valued at points in \mathcal{C}_0 , a critical point $c_{i,0}$ is considered periodic if there exists $n > 0$ such that $c_{i,0} \in T_0^n \{c_{i,0}\}$. Condition (IV) is necessary in order to ensure

that the transfer operators coresponding to the perturbed systems defined below satisfy uniform Lasota-Yorke inequalities. These uniform inequalities are essential for establishing the perturbative spectral results, Propositions 2 and 3, which are a key ingredient of our proof. Since we cannot exclude the possibility of the forward orbit of a critical point containing other critical points, these uniform inequalities do not follow directly from the original paper [10], but rather from later works [2, 1], see [7, Section 4.2].

In what follows, we consider C^2 -small perturbations $T_\varepsilon : I \curvearrowright$ of T_0 for $\varepsilon > 0$. A critical set for T_ε may be chosen as $\mathcal{C}_\varepsilon = \{0 = c_{0,\varepsilon} < c_{1,\varepsilon} < \dots < c_{d,\varepsilon} = 1\}$, where for each i , $\varepsilon \mapsto c_{i,\varepsilon}$ is a C^2 function for $\varepsilon \geq 0$. Furthermore, there exists $\delta > 0$ such that for all sufficiently small ε , there exists a C^2 extension $\hat{T}_{i,\varepsilon} : [c_{i,0} - \delta, c_{i+1,0} + \delta] \rightarrow \mathbb{R}$ of $T_\varepsilon|_{[c_{i,\varepsilon}, c_{i+1,\varepsilon}]}$, and $\hat{T}_{i,\varepsilon} \rightarrow \hat{T}_{i,0}$ in the C^2 topology.

We also assume:

(V) *The boundary points do not move, and no holes are created near them:* Precisely, for each $b \in \mathcal{B}$ we have the following

- (a) If $b \notin \mathcal{C}_0$, then necessarily $T_0(b) = b$. We assume further that for all $\varepsilon > 0$, $T_\varepsilon(b) = b$.
- (b) If $b \in \mathcal{C}_0$, we assume that $T_0(b_-) < b < T_0(b_+)$, and also that $b \in \mathcal{C}_\varepsilon$ for all ε .

This boundary condition can be considerably relaxed by suitably redefining the holes discussed below, as was explained in [7, Section 2.4]. For simplicity of presentation, we do not detail these generalizations here.

Set $H_{ij,\varepsilon} = I_i \cap T_\varepsilon^{-1}(I_j)$. We refer to these sets as *holes*. Once a T_ε -orbit enters a hole, it leaves one of the invariant sets for T_0 and continues in another. As $\varepsilon \rightarrow 0$, the holes converge (in the Hausdorff metric) to the infinitesimal holes from which they arise. Our assumptions imply that there exist numbers $\beta_{ij} \geq 0$ such that

$$\mu_i(H_{ij,\varepsilon}) = \varepsilon \beta_{ij} + o(\varepsilon). \quad (1)$$

Consider a continuous time Markov chain with states $1, 2, \dots, m$ and jump rates from state i to state j equal to β_{ij} .

(VI) *Irreducibility:* The Markov chain defined above is irreducible.

Condition **(VI)** implies that for small $\varepsilon > 0$, T_ε has only one ACIM μ_ε , with density $\phi_\varepsilon = d\mu_\varepsilon/dx$.

Examples of families T_ε that satisfy **(I)** through **(VI)** can be found in [7, Section 2.4].

It is natural to inquire if the family of ACIMs μ_ε has a unique limit as $\varepsilon \rightarrow 0$, and if this limit exists, how to express it as a convex combination of μ_1, \dots, μ_m . This problem was successfully addressed in [7].

Proposition 1 (Theorem 1 in [7]). As $\varepsilon \rightarrow 0$,

$$\phi_\varepsilon \xrightarrow{L^1} \phi_0 = \sum_j p_j \phi_j$$

where (p_1, p_2, \dots, p_m) is the invariant measure for the Markov chain.

We define μ_0 to be the measure whose density is ϕ_0 .

2.2 The main results

Proposition 1 shows that the limiting Markov chain provides useful information about our metastable system. In this section we prove two additional results making the connection between the dynamical system and the Markov chain more precise.

To state our first result we need to define, both for the T_ε -dynamics and for the Markov chain, a sequence of times that indicate when a transition occurs between different I_j 's.

For the Markov chain: Set $t_0^M = 0$, and for $i > 0$, let t_i^M be the i^{th} time that the Markov dynamics have changed states. Then for $i \geq 1$, set $\mathcal{T}_i^M = t_i^M - t_{i-1}^M$. Let z_i^M be the state of the chain after the i^{th} transition. Let \mathbb{P}^r denote the probability measure constructed on the space $\{1, \dots, m\}^{[0, \infty)}$ for the Markov chain by starting at I_r at $t = 0$ and then evolving forward by the Markov dynamics. Then, given the present state of the chain, \mathcal{T}_i^M are exponential random variables. That is, for $t \geq 0$, the densities are given by

$$d\mathbb{P}^r(\mathcal{T}_i^M = t | z_{i-1}^M = j) = \beta_j e^{-\beta_j t} dt$$

where $\beta_j = \sum_k \beta_{jk}$. Also given that $z_{i-1} = j$, z_i is independent of \mathcal{T}_i^M and

$$\mathbb{P}^r(z_i^M = k | z_{i-1}^M = j) = \beta_{jk} / \beta_j.$$

For T_ε : Let $z(x) = k$ if $x \in I_k$. Set $t_0^\varepsilon = 0$, and for $i > 0$, let $t_i^\varepsilon = \inf\{n > t_{i-1}^\varepsilon : z(T_\varepsilon^n x) \neq z(T_\varepsilon^{t_{i-1}^\varepsilon} x)\}$. Then for $i \geq 1$, set $\mathcal{T}_i^\varepsilon = t_i^\varepsilon - t_{i-1}^\varepsilon$.

Our first main result states that the finite dimensional distributions of jumps of the deterministic systems converge to the finite dimensional distributions of jumps of the Markov chain.

Theorem 1. Fix j, p and S . For any intervals $\Delta_k = [a_k, b_k]$, and numbers $r_k \in \{1, \dots, m\}$, $k = 1, \dots, p$

$$\mu_j(\varepsilon \mathcal{T}_k^\varepsilon \in \Delta_k, z(t_k^\varepsilon) = r_k, \text{ for } k = 1, \dots, p) \rightarrow \mathbb{P}^r(\mathcal{T}_k \in \Delta_k, z_k^M = r_k, \text{ for } k = 1, \dots, p)$$

and the convergence is uniform for $\max b_k \leq S$.

Consider an observable $A : I \rightarrow \mathbb{R}$. For each fixed small $\varepsilon > 0$, the Ergodic Theorem provides a law of large numbers, i.e. $N^{-1} \sum_{k=0}^N A \circ T_\varepsilon^k \rightarrow \mu_\varepsilon(A)$ a.e. and in L^1 as $N \rightarrow \infty$. If A is of bounded variation, the Central Limit Theorem applies [11] and states that $N^{-1/2} \sum_{k=0}^N A \circ T_\varepsilon^k$ approaches a normal distribution as $N \rightarrow \infty$. We let $D_\varepsilon(A)$ be the diffusion coefficient, which is the variance of the limiting normal distribution.

Theorem 2. For any observable A of bounded variation,

$$\varepsilon D_\varepsilon(A) \rightarrow \mathbf{D}^M(\mathbf{A})$$

as $\varepsilon \rightarrow 0$, where \mathbf{A} is the observable on the state space of our Markov chain such that $\mathbf{A}(j) = \int_{I_j} A \phi_j dx$ and \mathbf{D}^M stands for the diffusion coefficient.

Remark. \mathbf{D}^M can be computed efficiently. Assume for simplicity that \mathbf{A} has zero mean (otherwise we subtract a constant from \mathbf{A}). Let G denote the generator matrix of our Markov process, that is $G_{jk} = \beta_{jk}$ if $j \neq k$ and $G_{jj} = -\beta_j$. Then

$$\mathbf{D}^M = \sum_{jk} p_j \mathbf{A}(j) \int_0^\infty p_{jk}(t) \mathbf{A}(k) dt = \langle p \mathbf{A}, \int_0^\infty e^{tG} \mathbf{A} dt \rangle = \langle p \mathbf{A}, G^{-1} \mathbf{A} \rangle.$$

The proof of Theorems 1 and 2 depend on two perturbation results, Propositions 2 and 3. Proposition 3 is a variation on a result of Keller and Liverani [8, 9], see Appendix A for details, while Proposition 2 appears to be new. Our paper is organized as follows. In Section 3 we recall the transfer operator approach to the study of piecewise expanding interval maps and state Propositions 2 and 3. In Section 4 we derive Theorem 1 from Proposition 3. Section 5 contains the derivation of Theorem 2 from Theorem 1 and Proposition 2. Finally in Section 6 we explain how Proposition 2 follows from Theorem 1.

3 Preparatory material

3.1 Function spaces and norms

We use Leb to denote normalized Lebesgue measure on I and L^1 to denote the space of Lebesgue integrable functions on I . For $f : I \rightarrow \mathbb{R}$, let $|f|_{L^1} = \int_I |f(x)| dx$, $|f|_{L^\infty} = \sup_{x \in I} |f(x)|$, and $\text{var}(f)$ be the total variation of f over I ; that is,

$$\text{var}(f) = \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| : n \geq 1, 0 \leq x_0 < x_1 < \cdots < x_n \leq 1 \right\}.$$

$\text{BV} = \text{BV}(I)$ is the Banach space of functions $f : I \rightarrow \mathbb{R}$ with norm

$$\|f\| = \inf_{g=f \text{ except on a countable set}} \text{var}(g) + |g|_{L^\infty}.$$

An element of BV is technically an equivalence class of functions, any two of which agree except on a countable set; we generally ignore this distinction.

3.2 Transfer operators and the spectral setting

For $\varepsilon \geq 0$, let \mathcal{L}_ε be the transfer operator associated with T_ε acting on BV , i.e.

$$\mathcal{L}_\varepsilon A(x) = \sum_{y \in T_\varepsilon^{-1}\{x\}} \frac{A(y)}{|T'_\varepsilon(y)|}.$$

Note that \mathcal{L}_0 has 1 as an isolated eigenvalue of multiplicity m , and then a spectral gap. Let P denote the associate spectral projection, that is

$$(PA)(x) = \left(\int_{I_j} A(y) dy \right) \phi_j(x) \text{ if } x \in I_j.$$

For small $\varepsilon > 0$, 1 becomes an isolated eigenvalue creating a small spectral gap. Note that \mathcal{L}_ε preserves the space BV_0 of BV -functions with zero mean.

Proposition 2. There exists $\eta < 1$ and $\kappa > 0$ such that

$$\|\mathcal{L}_\varepsilon^{\kappa n/\varepsilon}\|_{\text{BV}_0} \leq \eta^n.$$

The proof of Proposition 2 is given in Section 6.

Remark. In case of two intervals much more precise asymptotics of the spectral gap can be deduced from the results of [9]. It is likely that a similar statement holds for an arbitrary number of intervals, however we do not pursue this question here since the weaker version stated above is sufficient for our purposes. In fact we hope that the argument used to prove Proposition 2 can be extended also to the case of piecewise hyperbolic systems such as billiards consisting of several regions connected by small holes.

For $j \in \{1, \dots, m\}$, let $\mathcal{L}_{j,\varepsilon}$ be the transfer operator acting on $\text{BV}(I_j)$ where we consider $H_{j,\varepsilon}$ as a hole, i.e.

$$\mathcal{L}_{j,\varepsilon}A(x) = \sum_{y \in (I_j \cap T_\varepsilon^{-1}\{x\})} \frac{A(y)}{|T'_\varepsilon(y)|}.$$

From [8] we see that, for small $\varepsilon > 0$, $\mathcal{L}_{j,\varepsilon}$ has an isolated simple eigenvalue $\lambda_{j,\varepsilon} < 1$ of multiplicity one with $\lambda_{j,\varepsilon} \rightarrow 1$ as $\varepsilon \rightarrow 0$, and otherwise the spectrum has a uniform spectral gap. More precisely, if $\mathcal{Q}_{j,\varepsilon}$ is the spectral projection corresponding to $\lambda_{j,\varepsilon}$, the following statement holds.

Proposition 3. There exists a measure $\nu_{j,\varepsilon}$ and a function $\phi_{j,\varepsilon} \in \text{BV}(I_j)$ such that

$$\nu_{j,\varepsilon}(\phi_{j,\varepsilon}) = 1, \quad \mathcal{Q}_{j,\varepsilon} = \nu_{j,\varepsilon}(\cdot)\phi_{j,\varepsilon}$$

and

- (a) $\lambda_{j,\varepsilon} = 1 - \beta_j\varepsilon + o(\varepsilon)$.
- (b) As $\varepsilon \rightarrow 0$, $\nu_{j,\varepsilon}(A) \rightarrow \int A(y)dy$ where the convergence holds in strong topology in BV^* and $\phi_{j,\varepsilon} \rightarrow \phi_j$ in L^1 . Moreover

$$\limsup_{\varepsilon \rightarrow 0} \sup_{H_{j,\varepsilon}} |\phi_{j,\varepsilon} - \phi_j| = 0. \tag{2}$$

- (c) There exists $C > 0$ such that, for all small ε , $A \in \text{BV}(I_j)$, and all $n \geq 0$,

$$\|\lambda_{j,\varepsilon}^{-n} \mathcal{L}_{j,\varepsilon}^n A - \mathcal{Q}_{j,\varepsilon} A\| \leq C\theta^n \|A\|.$$

This proposition follows from the work of Keller and Liverani [8, 9], see Appendix A for details.

4 Convergence of the jump process

4.1 Tightness

We need to know that the distribution of jumps is tight so we begin with the following result.

Lemma 1. Given S, δ, p there exists σ such that, for all ε sufficiently small,

$$\mu_p(\exists k \text{ with } t_k^\varepsilon \leq S/\varepsilon \text{ and } \mathcal{T}_{k+1}^\varepsilon \leq \sigma/\varepsilon) \leq \delta.$$

The following estimate plays a key role in our analysis. Given a segment $J \subset I$ let $r_n(x)$ be the distance of $T_\varepsilon^n x$ to the boundary of the component of $T_\varepsilon^n J$ containing it. Recall that Leb denotes Lebesgue measure on $I = [0, 1]$.

Lemma 2 (Growth Lemma). (see [3, Section 5.10]) There exists $\Lambda > 1$, $c > 0$ such that, for all ε small enough, and all J , and all $n \geq 0$,

$$\text{Leb}(x : r_n(x) \leq \varepsilon) \leq \text{Leb}(x : r_0(x) \leq \varepsilon/\Lambda^n) + c \text{Leb}(J)\varepsilon.$$

Proof of Lemma 1. First, $t_0^\varepsilon = 0$, and if $N = N(\sigma, \varepsilon) = \lceil \sigma/\varepsilon \rceil$, then

$$\mu_p(\mathcal{T}_1^\varepsilon \leq \kappa/\varepsilon) \leq 1 - \mu_p(\mathcal{T}_1^\varepsilon > N) = 1 - \int_L \mathcal{L}_{L,\varepsilon}^N(\phi_L) dx = 1 - \lambda_{p,\varepsilon}^N \int Q_{L,\varepsilon}(\phi_L) dx + O(\theta^N),$$

where we have used Proposition 3 (c). But from parts (a) and (b) of the same proposition, we see that this can be made arbitrarily small for all small ε by taking σ small enough.

Because $\mu_l \ll \text{Leb}$ it suffices to prove the statement for Leb .

We follow [5], Section 18. Let $S_{n,m} = \sum_{j=n+1}^{n+m} 1_{T_\varepsilon^j x \in H_\varepsilon}$. We have to show that

$$\sum_{n \leq S/\varepsilon} \int 1_{T_\varepsilon^n x \in H_\varepsilon} 1_{S_{n,\sigma/\varepsilon}(x) > 0} dx = o(1), \quad \varepsilon \rightarrow 0, \sigma \rightarrow 0.$$

Take a small r . We say that a visit of x to the hole at time n is (r -)inessential if the length of the smoothness component of $T_\varepsilon^n L \cap H_\varepsilon$ containing $T_\varepsilon^n x$ is less than $r\varepsilon$. By Lemma 2 the probability that x will have an inessential visit to the hole before time S/ε is less than $C r S$ which can be made as small as we wish by taking r small. Therefore it suffices to show that for any fixed r

$$\sum_{n \leq S/\varepsilon} \int 1_{\mathcal{E}_{n,\varepsilon}(x)} 1_{S_{n,\sigma/\varepsilon}(x) > 0} dx = o(1)$$

where $\mathcal{E}_{n,\varepsilon}(x) = \{x \text{ has essential visit to the hole at time } n\}$.

$$\int 1_{\mathcal{E}_{n,\varepsilon}(x)} 1_{S_{n,\sigma/\varepsilon}(x) > 0} dx = \text{Leb}(\mathcal{E}_{n,\varepsilon}(x)) \mathbb{P}(S_{n,\sigma/\varepsilon}(x) > 0 | \mathcal{E}_{n,\varepsilon}(x)).$$

Since $\text{Leb}(\mathcal{E}_{n,\varepsilon}(x)) \leq \text{Leb}(x : T_\varepsilon^n x \in H_\varepsilon) \leq C\varepsilon$ it suffices to check that

$$\max_{n \leq S/\varepsilon} \text{Leb}(S_{n,\sigma/\varepsilon} > 0 | \mathcal{E}_{n,\varepsilon}(x)) = o(1), \quad \varepsilon \rightarrow 0, \sigma \rightarrow 0. \quad (3)$$

In order to prove this we observe that due to assumption (II) for any fixed M_0 , $S_{n,M_0}(x) = 0$ for all $x \in \mathcal{E}_{n,\varepsilon}$ provided that ε is small enough. On the other hand if $k \geq M_0$ then by Lemma 2 applied to H_ε

$$\text{Leb}(1_{H_\varepsilon} T_\varepsilon^{n+k} x) > 0 | (\mathcal{E}_{n,\varepsilon}(x)) \leq C (\varepsilon + \Lambda^{-k}).$$

Summing over $k \in [M_0 + 1, \kappa/\varepsilon]$ we obtain (3). \square

4.2 Convergence of the finite dimensional distributions

Proof of Theorem 1. The proof is by induction on p . First, for $p = 1$ assume that the initial distribution of x is chosen according to some density $\rho \in BV(\mu_j)$. Then by Proposition 3(c)

$$\mu_j(\mathcal{T}_1^\varepsilon = n, z(T_\varepsilon^n x) = r) = \int_{H_{jr}} \mathcal{L}_{j,\varepsilon}^n(\rho) dx = \lambda_{j,\varepsilon}^n \int_{H_{jr}} (\mathcal{Q}_{j,\varepsilon} \rho) dx + O(\theta^n).$$

Recall that

$$\mathcal{Q}_{j,\varepsilon} \rho = \nu_{j,\varepsilon}(\rho) \phi_{j,\varepsilon}.$$

By Proposition 3 $\nu_{j,\varepsilon}(\rho) \rightarrow \int \rho(x) dx$, $\int_{H_{jr}} \phi_{j,\varepsilon}/\varepsilon \rightarrow \beta_{jr}$ and for $\varepsilon n \approx t$, $\lambda_{j,\varepsilon}^n \rightarrow e^{-\beta_j t}$. Thus summation over $n \in \Delta/\varepsilon$ concludes the proof for $p = 1$.

Next suppose that the statement is known for some p . Denote

$$\Omega = \{\mathcal{T}_k^\varepsilon \in \Delta_k, z(t_k^\varepsilon) = r_k, \text{ for } k = 1, \dots, p\}.$$

To carry the induction step it is enough to prove that

$$\mu_j(\varepsilon \mathcal{T}_{p+1}^\varepsilon \in \Delta_{p+1}, z(t_{p+1}^\varepsilon) = r_{p+1}, \Omega) \rightarrow \mu_j(\Omega) \beta_{z_p z_{p+1}} \int_{\Delta_{p+1}} e^{-\beta_{z_p} t} dt. \quad (4)$$

Set

$$(\mathcal{L}_\Omega A)(x) = \sum_{T_\varepsilon^{t_\varepsilon^p} y = x} \frac{A(y)}{(T_\varepsilon^{t_\varepsilon^p})'(y)}$$

where the sum is taken over $y \in \Omega$. Then

$$\mu_j((\varepsilon \mathcal{T}_{p+1}^\varepsilon = n, z(t_{p+1}^\varepsilon) = r_{p+1}, \Omega) = \int_{H_{z_p z_{p+1}}} \mathcal{L}_{j,\varepsilon}^n(\mathcal{L}_\Omega(1_{I_j} \phi_j)) dx. \quad (5)$$

Take a small σ and rewrite

$$\mathcal{L}_{j,\varepsilon}^n(\mathcal{L}_\Omega(1_{I_j} \phi_j)) = \mathcal{L}_{j,\varepsilon}^{n-\sigma/\varepsilon} \left[\mathcal{L}_{j,\varepsilon}^{\sigma/\varepsilon}(\mathcal{L}_\Omega(1_{I_j} \phi_j)) \right].$$

Due to Lasota-Yorke inequality

$$\left[\mathcal{L}_{j,\varepsilon}^{\sigma/\varepsilon}(\mathcal{L}_\Omega(1_{I_j} \phi_j)) \right]$$

has bounded BV-norm. Hence arguing as in the $p = 1$ case we see that (5) is asymptotic to

$$\beta_{z_p z_{p+1}} \int_{\Delta_{p+1}} e^{-\beta_{z_p} t} dt \int_I \left[\mathcal{L}_{j,\varepsilon}^{\sigma/\varepsilon}(\mathcal{L}_\Omega(1_{I_j} \phi_j)) \right] dx (1 + o_{\sigma \rightarrow 0}(1)).$$

Due to Lemma 1 the last integral here equals to

$$\mu_j(\Omega, \mathcal{T}_{p+1}^\varepsilon > \sigma/\varepsilon) = \mu_j(\Omega) + o_{\sigma \rightarrow 0}(1).$$

Since σ is arbitrary this proves (4) completing the proof of Theorem 1. \square

5 Proof of Theorem 2

In this section we deduce Theorem 2 from Theorem 1 and Proposition 2.

Proof. Let $\bar{A} = A - \mu_\varepsilon(A)$. By Proposition 2 for each δ we can find S such that

$$\left| \sum_{n \geq S/\varepsilon} \mu_\varepsilon(\bar{A}\bar{A} \circ T_\varepsilon^n) \right| < \delta$$

so it suffices to get the asymptotics of $\mu_\varepsilon(\bar{A}\bar{A} \circ T_\varepsilon^n)$ for $n \approx t/\varepsilon$ where $t \leq S$. We need to estimate $D_n = \mu_\varepsilon(\bar{A}\bar{A} \circ T_\varepsilon^n)$. Take a natural number n_0 . We have

$$D_n = \int \bar{A}(x) \phi_\varepsilon(x) \bar{A}(T_\varepsilon^n x) dx = \int \bar{A}(T_\varepsilon^{n_0} y) [\mathcal{L}_\varepsilon^{n-n_0}(\mathcal{L}_\varepsilon^{n_0}(\bar{A}\phi_\varepsilon))](y) dy.$$

Since L_ε depends continuously on ε as a map $BV \rightarrow L^1$ we can rewrite the last expression as

$$\int \bar{A}(T_\varepsilon^{n_0} y) [\mathcal{L}_\varepsilon^{n-n_0}(P(\bar{A}\phi_\varepsilon))](y) dy + O(\theta^{n_0}) + o_{\varepsilon \rightarrow 0}(1).$$

Since T_0 is mixing we have

$$\int \bar{A}(T_\varepsilon^{n_0} y) B(y) dy = \sum_k \left(\int_{I_k} \bar{A}(y) \phi_k dy \int_{I_k} B(y) dy \right) + [O(\theta^{n_0}) + o_{\varepsilon \rightarrow 0}(1)] \|A\|_{BV} \|B\|_{BV}.$$

As $\varepsilon \rightarrow 0$ the first factor converges to $\mathbf{A}(k) - \sum_l p_l \mathbf{A}(l)$ while the second term equals

$$\begin{aligned} \int_{I_k} \mathcal{L}_\varepsilon^{n-2n_0} P(\bar{A}\phi_\varepsilon p) &= \sum_j 1_{I_k} p_j \left(\mathbf{A}(j) - \sum_l p_l \mathbf{A}(l) \right) \mathcal{L}_\varepsilon^{n-2n_0} \phi_j dy + o_{\varepsilon \rightarrow 0}(1) \\ &= \sum_j p_j \left(\mathbf{A}(j) - \sum_l p_l \mathbf{A}(l) \right) \mu_j(T_\varepsilon^{n-2n_0} x \in I_k). \end{aligned}$$

Since n_0 was arbitrary we conclude that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \sum_{n \leq S/\varepsilon} \mu_\varepsilon(\bar{A}\bar{A} \circ T_\varepsilon^n) = \sum_{jk} \int_0^S \left[p_j \mathbf{A}(j) \mathbf{A}(k) p_{jk}(t) - \left(\sum_j p_j \mathbf{A}(j) \right)^2 \right] dt.$$

Since S is arbitrary we can let $S \rightarrow \infty$ and obtain Theorem 2. \square

6 Proof of Proposition 2

Proof. Recall that \mathcal{L}_ε satisfy uniform Lasota-Yorke Inequality, that is there is a constant K such that

$$\text{var}(\mathcal{L}_\varepsilon^n A) \leq K [\|A\|_{L^1} + \theta^n \text{var}(A)] \quad (6)$$

$$\|\mathcal{L}_\varepsilon^n A\|_{L^1} \leq K \|A\|_{L^1} \quad (7)$$

Consequently it suffices to show that if $A \in \text{BV}_0$ then

$$\|\mathcal{L}_\varepsilon^{\kappa/2\varepsilon} A\|_{L^1} \leq (4K)^{-1} \|A\|_{\text{BV}} \quad (8)$$

since combining (6), (7) and (8) gives

$$\|\mathcal{L}_\varepsilon^{\kappa/\varepsilon} A\|_{\text{BV}} \leq 2^{-1} \|A\|_{\text{BV}}.$$

Next mixing of T_0 on I_k implies that if $\int_{I_k} A(x)dx = 0$ for each k then

$$\|\mathcal{L}_0^{n_0} A\|_{L^1} \leq C\theta^{n_0} \|A\|_{\text{BV}}.$$

Consequently given δ we can find n_0, ε_0 and δ_1 such that if $\varepsilon \leq \varepsilon_0$ and $|\int_{I_k} A(x)dx| \leq \delta_1$ for each k then

$$\|\mathcal{L}_\varepsilon^{n_0}\|_{L^1} \leq \delta.$$

Let $\bar{n} = \frac{\kappa}{2\varepsilon} - n_0$. By the foregoing discussion it remains to show that for each k

$$\left| \int_{I_k} (\mathcal{L}_\varepsilon^{\bar{n}} A)(x)dx \right| \leq \delta_1$$

provided that κ is large enough. Since

$$\mathcal{L}_\varepsilon^{\bar{n}} A = \mathcal{L}_\varepsilon^{\bar{n}-n_1} \mathcal{L}_\varepsilon^{n_1} A = \mathcal{L}_\varepsilon^{\bar{n}-n_1} (PA) + O(\theta^{n_1}) + o_{\varepsilon \rightarrow 0}(1)$$

we need to show that $\int_{I_k} (\mathcal{L}_\varepsilon^{\bar{n}-n_1} PA)(x)dx$ is small. By Theorem 1 this integral is asymptotic to $\sum_j \int_{I_j} A(y)dy p_{jk}(\kappa)$. As $\kappa \rightarrow \infty$ this expression converges to $p_k \int_I A(y)dy = 0$ so the result follows. \square

A Proof of Proposition 3

Proof. Part (a) is proven using arguments similar to the ones in [9, Section 3.2]. Namely we apply [9, Theorem 2.1] with $\mathcal{P}_0 = \mathcal{L}_0$, $\mathcal{P}_\varepsilon = \mathcal{L}_{j,\varepsilon}$, and the Banach space $\text{BV}(I_j)$. This theorem says that

$$\lim_{\varepsilon \rightarrow 0} \frac{1 - \lambda_{j,\varepsilon}}{\Delta_\varepsilon} = 1 - \sum_{k \geq 0} q_k$$

with

$$\begin{aligned} \Delta_\varepsilon &= \mu_j(H_{j,\varepsilon}) = \beta_j \varepsilon + o(\varepsilon), \\ q_k &= \lim_{\varepsilon \rightarrow 0} \frac{\mu_j(T_0^{-1} T_\varepsilon^{-k} H_{j,\varepsilon}) - \mu_j(T_\varepsilon^{-(k+1)} H_{j,\varepsilon})}{\Delta_\varepsilon}. \end{aligned}$$

Now for fixed k and all small ε the set $T_\varepsilon^{-k} H_{j,\varepsilon}$ consists of a finite number of intervals of size $O(\varepsilon)$ so the sizes of their preimages by T_0 and T_ε differ by $O(\varepsilon^2)$ (here we are using assumption **(II)**, which implies that ϕ_0 is continuous at points in $T_0^{-(k+1)} H_\varepsilon$).

This completes the proof of part (a). Parts (b) and (c) follow from [8] except for (2) which is proven in [7]. \square

References

- [1] V. Baladi. *Positive transfer operators and decay of correlations*, volume 16 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000.
- [2] V. Baladi and L.-S. Young. On the spectra of randomly perturbed expanding maps. *Comm. Math. Phys.*, 156(2):355–385, 1993.
- [3] Chernov N., Markarian R. *Chaotic billiards*, Math. Surv. & Monogr. **127** AMS, Providence, RI, 2006.
- [4] Demers M. F., Young Lai-Sang *Escape rates and conditionally invariant measures* Nonlinearity **19** (2006) 377–397.
- [5] Dolgopyat D. *Limit Theorems for partially hyperbolic systems*, Trans. AMS **356** (2004) 1637–1689.
- [6] Freidlin M. I., Wentzell A. D. *Random perturbations of dynamical systems*, 2d ed. Grundlehren der Mathematischen Wissenschaften **260** (1998) Springer, New York.
- [7] Gonzalez Tokman C., Hunt B., Wright P. *Approximating invariant densities of metastable systems*, preprint.
- [8] Keller G., Liverani C. *Stability of the spectrum for transfer operators*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze **XXVIII** (1999) 141–152.
- [9] Keller G., Liverani C. *Rare events, escape rates and quasistationarity: some exact formulae*, Journal of Statistical Physics **135** (2009), 519–534.
- [10] A. Lasota and J.A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.*, 186:481–488, 1973.
- [11] Liverani C. *Central limit theorem for deterministic systems*, in International Conference on Dynamical Systems (Montevideo, 1995), 56–75, Pitman Res. Notes Math. Ser., **362** Longman, Harlow, 1996.
- [12] Machta J., Zwanzig R. *Diffusion in a periodic Lorentz gas*, Phys. Rev. Lett. **50** (1983) 1959–1962.